# Lecture 15.

- Finish up Möbius transf.

**Recall.** A Möbius tranformation sends circles on $\mathbb{C}_\infty$ to circles on $\mathbb{C}_\infty$.

**Thm 1.** Given circles $\Gamma, \Gamma' \subseteq \mathbb{C}_\infty$, $\exists$ Möbius $T$ s.t. $T(\Gamma) = \Gamma'$. Moreover, given $z_1, z_2, z_3 \in \Gamma$, $w_1, w_2, w_3 \in \Gamma'$, $\exists$ unique $T(\Gamma) = \Gamma'$ w/ $T(z_j) = w_j$, $j = 1, 2, 3$.

**Pf.** DIY.

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## Complex Integration

**Recall.** A $C^1$ (or smooth) path in $G \subseteq \mathbb{C}$ is a smooth map $\gamma : [a,b] \to \mathbb{C}$ s.t. $\gamma' \neq 0$. A piecewise (p.w.) smooth path is a continuous map with a finite partition $a = a_0 < a_1 < \dots < a_n = b$ s.t. $\gamma|_{[a_{j-1}, a_j]}$ is smooth.

In this course, <u>only</u> p.w. smooth paths will be considered. Functions of bounded variation and rectifiable paths will not be part of material.

<u>Def.</u> (1) If $f$ is continuous in $G$ and $\gamma : [a,b] \to G$ a smooth path, then

- $$\int_\gamma f\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt. \quad \left(\begin{matrix}\text{path}\\ \underline{\text{integral}}\end{matrix}\right)$$

- Length of $\gamma$, $\int_\gamma |dz| = \int_a^b |\gamma'(t)|\,dt$.

<u>Prop!</u> If $f$ is cont. in $G$, $\gamma : [a,b] \to G$ smooth path, $\varphi : [c,d] \to [a,b]$ smooth surjective map w/ $\varphi' \geq 0$, then $\sigma = \gamma \circ \varphi$ is a smooth path and

$$\int_\sigma f\,dz = \int_\gamma f\,dz.$$

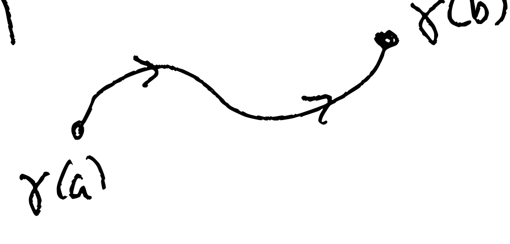
Pf. Change of variables formula:

$$\int_\gamma f\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt = \left\{ \begin{matrix} t = \varphi(s) \\ dt = \varphi'(s)\,ds \end{matrix} \right\}$$

$$= \int_c^d f(\gamma\circ\varphi(s))\,\gamma'(\varphi(s))\,\varphi'(s)\,ds = \{\sigma = \gamma\circ\varphi\}$$

$$= \int_c^d f(\sigma(s))\,\sigma'(s)\,ds = \int_\sigma f\,dz. \quad \blacksquare$$

A map $\varphi : [c,d] \to [a,b]$ as in Prop 1 is a reparametrization of $\gamma$. The paths $\gamma$ and $\sigma = \gamma\circ\varphi$ are equivalent. Equivalence classes of paths are called curves. A path integral depends only on the curve, which we think of as the image $\{\gamma\} \subseteq G$ together w/ a direction (orientation)



<u>Prop 2.</u> Given a smooth curve $\gamma : [a,b] \to G$ $\exists$ repar. $\varphi : [0,L] \to [a,b]$ s.t. $\sigma = \gamma \circ \varphi$ satisfies $|\sigma'| = 1$ and $L = \int_\gamma |dz|$.

<u>Pf:</u> Define $s(t) := \int_a^t |\gamma'(r)|\,dr$, then

$L = \int_\gamma |dz|$, $s : [a,b] \to [0,L]$ is $C^1$, surjective and $s'(t) = |\gamma'(t)| > 0$.

Thus, $s$ has an inverse $\varphi : [0,L] \to [a,b]$ $t = \varphi(s(t))$ s.t. $1 = \varphi'(s(t))\, s'(t) \Rightarrow$

$\varphi'(s) = \dfrac{1}{|\gamma'(\varphi(s))|}$. Then, w/ $\sigma = \gamma \circ \varphi$

$$\sigma'(s) = \gamma'(\varphi(s)) \frac{1}{|\gamma'(\varphi(s))|} \Rightarrow$$

$$|\sigma'(s)| = 1 \text{ as desired. } \boxtimes$$

<u>Rem.</u> The path $\sigma : [0,L] \to G$ is called the <u>arclength param.</u> of the curve.

Path integrals have all the usual props of integrals, including if $a = a_0 < a_1 < \dots < a_n = b$ is a partition, then

$$\int_{\gamma} f\,dz = \sum_{j=1}^{n} \int_{\gamma_j} f\,dz,$$

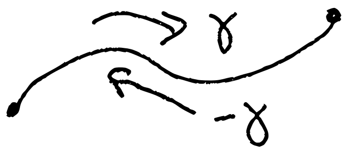
where $\gamma_j = \gamma\big|_{[a_{j-1}, a_j]}$.

<u>Def.</u> (2) If $\gamma : [a,b] \to G$ is p.w. smooth and $a = a_0 < \dots < a_n = b$ is a partition s.t. $\gamma_j : [a_{j-1}, a_j] \to G$ is smooth, then

$$\int_{\gamma} f\,dz = \sum_{j=1}^{n} \int_{\gamma_j} f\,dz.$$

The property above $\Rightarrow$ this definition is indep. of partition and, hence, def. is consistent.

We also introduce $-\gamma$ to be the curve corresponding to $\gamma$ "traveling the opposite direction":




$-\gamma$ is parametrized by, e.g.,
$(-\gamma)(t) = \gamma((1-t)b + ta)$ if $\gamma : [a,b] \to G$
parametrizes the curve $\gamma$. It is easy to see that

$$\int_{-\gamma} f\,dz = -\int_{\gamma} f\,dz.$$

## "Fundamental Thm of Calculus"

Let $f$ be cont. in $G \subseteq \mathbb{C}$ and assume $\exists F$ in $G$ w/ $F' = f$. Then, if $\gamma$ is curve from $z_0 \in G$ to $z_1 \in G$,

$$\int_{\gamma} f\,dz = F(z_1) - F(z_0)$$

**Pf.** DIY. It is truly the FTC. ☒

**Rem.** $F$ in FTC is analytic. Thus, only "special" $f$ can have such a primitive fcn. As we shall see, $f$ is also analytic. (We know this is true if $F$ were a power series.) But even for analytic $f$, $F$ need not exist. Depends on topology of $G$.

**Ex.** (1) Let $G = \mathbb{C} \setminus \{0\}$ and $f(z) = 1/z$. If $F$ existed, it must be a branch of $\log z$ + constant. But $\log z$ has no branch in $\mathbb{C} \setminus \{0\}$!